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LETTER TO THE EDITOR

Yang–Baxter matrix and ∗-calculi on quantum groups of A-series

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Abstract. In this letter, we prove that the differential calculi on quantum groups of A-series given in this letter and in previous papers are ∗-calculi.

It is well known that from the Yang–Baxter matrix

$$R_q = q^{1/N}\left(\sum_{i,j=1}^{N} q^{\delta_{ij}} e_{ii}\otimes e_{jj} + (q-q^{-1})\sum_{\substack{i,j=1\\ i>j}}^{N} e_{ij}\otimes e_{ji}\right) \qquad q\in\mathbb{C}^* \tag{1}$$

we can construct a quantum group of A-series [1]. The coordinate ring of $GL_q(N)$ is generated on $\mathbb{C}$ by $(\mathrm{Det}_q T)^{-1}$, the inverse element of quantum determinant $\mathrm{Det}_q T$ and t_{ij} $(i,j = 1,2,\ldots,N)$ which satisfy the relation

$$R_q T_1 T_2 = T_2 T_1 R_q\,. \tag{2}$$

If the relation $\mathrm{Det}_q T = 1$ is added, we obtain the quantum group $SL_q(N)$.

Recently, differential calculi on quantum planes and quantum groups have been discussed in many papers: Wess and Zumino gave the general methods to study differential calculi on quantum planes [2], Woronowicz provided basic theory of differential calculi on quantum groups [3] and many other people gave methods to construct differential calculi on quantum groups (such as [4] and [5]), but the ∗-calculi have not been discussed very much. In this letter, we give the differential calculi on the quantum group $SU_q(N)$ as an extension of the results of [6–8] and prove that these differential calculi and the differential calculus given in [9] are ∗-calculi. We will point out that the Yang–Baxter matrix plays a very important role in the ∗-calculi on quantum groups.

Let Ω^0 be the coordinate ring of the A-series quantum group $\mathcal{A}$. The first-order differential calculus is denoted by $\{\Omega^1, d\}$, where Ω^1 is a bimodule of Ω^0 and d is a linear operator from Ω^0 to Ω^1 satisfying:

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(i) Leibnitz rule $d(xy) = (dx)y + x\,dy$, $\forall x, y \in \Omega^0$,

(ii) for an arbitrary element ρ in Ω^1, there always exist some elements $x_k, y_k \in \mathcal{A}$ $(k = 1, 2, \ldots, M)$ such that $\rho = \sum_{k=1}^{M} x_k\, dy_k$.

On the quantum group of A-series, we have two sets of Y–B linear functionals $L^{\pm} = (l_{ij}^{\pm})_{1 \leqslant i,j \leqslant N}$ defined by

$$\langle L^+, T\rangle = \lambda_+ P R_q P \qquad \langle L^-, T\rangle = \lambda_-^{-1} R_q^{-1} \qquad \lambda_{\pm} \in \mathbb{C}^* \tag{3}$$

where P is the permutation matrix. If $\mathcal{A}$ denotes the quantum group $SL_q(N)$, we must require $\lambda_+^N = \lambda_-^N = 1$. Furthermore, if we introduce two sets of functionals on Ω^0 as follows:

$$\nabla_{ij} = \frac{1}{q - q^{-1}}(S(l_{ik}^-)l_{kj}^+ - \delta_{ij}\varepsilon) \tag{4}$$

$$\theta_{ijkl} = S(l_{ki}^-)l_{jl}^+ \tag{5}$$

where $i, j, k, l = 1, 2, \ldots, N$, S is the antipode, then we have:

Proposition 1.1. For $\forall x, y \in \Omega^0$, $i, j, k, l, u, v = 1, 2, \ldots, N$, we have:

(i) $\nabla_{ij}(\mathbf{1}) = 0$, $\theta_{ijkl}(1) = \delta_{ik}\delta_{jl}$,

(ii) $\Delta\nabla_{ij} = \nabla_{uv} \otimes \theta_{uvij} + \varepsilon \otimes \nabla_{ij}$, $\Delta\theta_{ijkl} = \theta_{ijuv} \otimes \theta_{uvkl}$,

(iii) $\nabla_{ij} * (xy) = (\nabla_{uv} * x)(\theta_{uvij} * y) + x(\nabla_{ij} * y)$, $\theta_{ijkl} * (xy) = (\theta_{ijuv} * x)(\theta_{uvkl} * y)$.

The proof of proposition 1.1 can be found in [7]. Let Ω^1 be the left module generated by ω^{ij} $(i, j = 1, 2, \ldots, N)$; therefore the first-order differential calculus on $\mathcal{A}$ is given by

$$dx = (\nabla_{ij} * x)\omega^{ij} \tag{6}$$

$$\omega^{ij} \cdot x = (\theta_{ijkl} * x)\omega^{kl} \qquad \forall x \in \Omega^0 \quad i, j, k, l = 1, 2, \ldots, N\,. \tag{7}$$

From the discussion in [8], we know (6) and (7) in fact give the first-order bicovariant differential calculus on the quantum group of A-series. Furthermore the quantum de Rham complex on $\mathcal{A}$ is defined by

$$\Omega^{\wedge} = \Omega^{\otimes}/\{\ker(1 - \sigma)\} \tag{8}$$

where

$$\ker(1 - \sigma) = \left[(\mathbf{R} + q^2 E_{N^4})(\mathbf{R} + q^{-2} E_{N^4})\right]_{ijkl}^{\alpha\beta\gamma\delta} \omega^{ij} \otimes \omega^{kl} \qquad \alpha, \beta, \gamma, \delta, i, j, k, l = 1, 2, \ldots, N \tag{9}$$

and

$$\mathbf{R} = (PR_q^{t_1})_{23}(R_q^t P)_{12}(PR_q^{-1})_{34}(PR_q^{t_1})_{23}^{-1}\,. \tag{10}$$

From (9) and (10), it can be seen that the Y–B matrix plays a very important role in the construction of the quantum de Rham complex.

Based on the property of the Y–B matrix R_q of A-series, we have the following important proposition:

Proposition 1.2. Let ∇_{ij} and θ_{ijkl} $(i,j,k,l = 1,2,\ldots,N)$ be defined by (4) and (5). We have

$$\nabla_{ij}(t_{ab}) = \nabla_{ji}(t_{ba}) \qquad \theta_{ijkl}(t_{ab}) = \theta_{jilk}(t_{ba}) \qquad i,j,k,l,a,b = 1,2,\ldots,N\,.$$

Remark:. For the definition of bicovariant differential calculus on quantum groups, see [3].

If we introduce an operator $*$: $\Omega^0 \longrightarrow \Omega^0$ to $SL_q(N)$ satisfying

$$T^* = S(T)^t \tag{11}$$

where S is the antipode, then the quantum group $SU_q(N)$ is obtained. If the $*$ operator can be extended to an operator on the quantum de Rham complex $\Omega^\wedge$ satisfying

$$(\rho_1 \wedge \rho_2)^* = (-1)^{k_1 k_2}\rho_2^* \wedge \rho_1^* \qquad (d\rho)^* = d(\rho^*) \qquad \rho,\rho_1,\rho_2 \in \Omega^\wedge \tag{12}$$

where k_1 and k_2 are the orders of ρ_1 and ρ_2 respectively, then (Ω^1, d) is called a $*$-calculus. According to the basic theory of the quantum matrix group of Woronowicz [3], (Ω^1, d) is a $*$-calculus $\Longleftrightarrow$ $S(\mathcal{H})^* \subseteq \mathcal{H}$, where $\mathcal{H} = \ker\varepsilon \cap \{\cap_{i,j=1}^N \ker\nabla_{ij}\}$, ε is the co-unit.

By [8], if we denote the set of the generators of the right ideal $\mathcal{H}$ by Λ, then for the quantum group $SU_q(N)$, the elements of Λ can be written as

$$\xi_{abcd} = t_{ab}t_{cd} - \nabla_{ij}(t_{ab}t_{cd})(M^{-1})^{ij}_{kl}t_{kl} - C_{abcd}$$

where

$$M^{ij}_{kl} = \nabla_{kl}(t_{ij}) \qquad C_{abcd} = \varepsilon(t_{ab}t_{cd} - \nabla_{ij}(t_{ab}t_{cd})(M^{-1})^{ij}_{kl}t_{kl})\,.$$

By (11), we know

$$S(t_{ij})^* = t_{ji}\,.$$

By proposition 1.2, we have

$$\nabla_{ij}(t_{ab}t_{cd}) = \theta_{ijkl}(t_{ab})\nabla_{kl}(t_{cd}) - \delta_{ab}\nabla_{ij}(t_{cd}) = \nabla_{ji}(t_{ba}t_{dc})$$
$$(M^{-1})^{ij}_{kl} = (M^{-1})^{ji}_{lk}\,.$$

Therefore

$$S(\xi_{abcd})^* = \xi_{badc}$$

i.e. $S(\Lambda)^* = \Lambda$, and then we straightforwardly have $S(\mathcal{H})^* \subseteq \mathcal{H}$. Hence we have proved that the differential calculus on $SU_q(N)$ is a $*$-calculus. Therefore we have N differential $*$-calculi on $SU_q(N)$ different from each other by the choice of the product $r = \lambda_+\lambda_-$ $(r^N = 1)$.

The construction of the quantum Lorentz group was first given by Podlés and Woronowicz [10], and then discussed in some other papers. In fact the quantum Lorentz group can be treated as $SL_q(2,\mathbb{C})$, the complex version of $SL_q(2)$. The coordinate ring of a quantum Lorentz group is generated by 8 elements t_{ij} and $t_{\bar{i}\bar{j}}$ $(i,j=1,2)$. If we arrange them into two 2×2 matrices as

$$\mathcal{T}=\begin{pmatrix} T & \\ & \hat{T}\end{pmatrix} \qquad T=(t_{ij})_{i,j=1,2} \qquad \hat{T}=(t_{\bar{i}\bar{j}})_{i,j=1,2}$$

then the relations satisfied by these elements can be written as

$$t_{\bar{i}\bar{j}}=(S(t_{ji}))^* \qquad \mathrm{Det}_q T=t_{11}t_{22}-qt_{12}t_{21}=1 \qquad \mathrm{Det}_q\hat{T}=t_{\bar{1}\bar{1}}t_{\bar{2}\bar{2}}-qt_{\bar{1}\bar{2}}t_{\bar{2}\bar{1}}=1$$

and

$$\mathcal{R}\mathcal{T}_1\mathcal{T}_2=\mathcal{T}_2\mathcal{T}_1\mathcal{R} \tag{13}$$

where $\mathcal{R}=(\mathcal{R}^{ab}_{cd})_{a,b,c,d=1,2,\bar{1},\bar{2}}$,

$$\mathcal{R}^{ij}_{kl}=R^{ij}_{kl} \qquad \mathcal{R}^{i\bar{j}}_{k\bar{l}}=((R^+)^{-1})^{ij}_{kl} \qquad \mathcal{R}^{\bar{i}\bar{j}}_{\bar{k}\bar{l}}=R^{ij}_{kl} \qquad \mathcal{R}^{\bar{i}j}_{\bar{k}l}=((R^+)^{-1})^{ij}_{kl}$$
$$i,j,k,l=1,2$$

and R is 2×2 Y–B matrix of A-series and other elements of $\mathcal{R}$ are zeros. It can be checked that $\mathcal{R}$ also satisfies the Yang–Baxter equation.

We can define two sets of linear functionals by

$$\langle l^+_{ab},t_{cd}\rangle=P\mathcal{R}P \qquad \langle l^-_{ab},t_{cd}\rangle=\mathcal{R}^{-1} \qquad a,b,c,d=1,2,\bar{1},\bar{2}\,.$$

We can also define ∇_{ab} and θ_{abcd} $(a,b,c,d=1,2,\bar{1},\bar{2})$ by (4) and (5) as we have done for $SU_q(N)$.

The differential calculus on quantum Lorentz was discussed in [9], and the elements of the generators set Λ corresponding to the quantum Lorentz group can be written as

$$\xi_{abcd}=t_{ab}t_{cd}-\nabla_{ij}(t_{ab}t_{cd})(M^{-1})^{ij}_{kl}t_{kl}-C_{abcd}$$
$$\xi_{\bar{a}\bar{b}\bar{c}\bar{d}}=t_{\bar{a}\bar{b}}t_{\bar{c}\bar{d}}-\nabla_{\bar{i}\bar{j}}(t_{\bar{a}\bar{b}}t_{\bar{c}\bar{d}})(M^{-1})^{\bar{i}\bar{j}}_{\bar{k}\bar{l}}t_{\bar{k}\bar{l}}-C_{\bar{a}\bar{b}\bar{c}\bar{d}}$$

where

$$M^{ij}_{kl}=\nabla_{kl}(t_{ij}) \qquad M^{\bar{i}\bar{j}}_{\bar{k}\bar{l}}=\nabla_{\bar{k}\bar{l}}(t_{\bar{i}\bar{j}})$$
$$C_{abcd}=\varepsilon(t_{ab}t_{cd}-\nabla_{ij}(t_{ab}t_{cd})(M^{-1})^{ij}_{kl}t_{kl})$$
$$C_{\bar{a}\bar{b}\bar{c}\bar{d}}=\varepsilon(t_{\bar{a}\bar{b}}t_{\bar{c}\bar{d}}-\nabla_{\bar{i}\bar{j}}(t_{\bar{a}\bar{b}}t_{\bar{c}\bar{d}})(M^{-1})^{\bar{i}\bar{j}}_{\bar{k}\bar{l}}t_{\bar{k}\bar{l}})\,.$$

After some computation we have

$$\nabla_{ij}(t_{ab}t_{cd})(M^{-1})^{ij}_{kl}=\nabla_{\bar{i}\bar{j}}(t_{\bar{a}\bar{b}}t_{\bar{c}\bar{d}})(M^{-1})^{\bar{i}\bar{j}}_{\bar{k}\bar{l}}$$

and

$$S(\xi_{abcd})^*=\xi_{\bar{b}\bar{a}\bar{d}\bar{c}} \qquad S(\xi_{\bar{a}\bar{b}\bar{c}\bar{d}})^*=\xi_{badc}\,.$$

Therefore, $S(\Lambda)^*=\Lambda$, and $S(\mathcal{H})^*\subseteq\mathcal{H}$. Thus we have proved that the differential calculus on the quantum Lorentz group given in [9] is a $*$-calculus.

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